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SUMMARY

A perturbation technique is used to obtain approximate solutions to the nonlinear hydromagnetic equations (Ohm's law and the equation of motion of a fluid element) of a cold collisionless plasma. The assumed model consequently consists of (1) a linear harmonic wave that is infinite in extent and (2) nonlinear waves that are harmonic and propagate in the direction of the fundamental. The perturbation technique limits application to regions where nonlinearity is small. Some idea of the importance of the nonlinearity can be obtained by observing the magnitude of the terms. For a transverse perpendicularly propagating wave in atomic hydrogen, nonlinear effects are shown to be important near and below a frequency of 10^7 second^{-1} for a fundamental amplitude of 1 statvolt per centimeter or greater and for a plasma density of 10^{12} ions per cubic centimeter and a magnetic field intensity of 10^4 gauss. For a longitudinal perpendicularly propagating or a right or left circularly polarized wave, nonlinear effects are only important for very large wave amplitudes.

INTRODUCTION

Nonlinear calculations for plasma wave propagation are generally very difficult because of the complexity of the hydromagnetic equation (Ohm's law and the equation of motion of a fluid element). Consequently, to make the problem tractable, simplifying assumptions must be made. One such heuristic hypothesis entails assuming a specific structure for the nonlinear plasma wave. Experimental characteristics can be used as a guide to establish the a priori structure. One typical characteristic of experimental nonlinear steady-state systems is that numerous discrete frequencies in addition to the frequency of the forcing oscillation (pp. 81 and 82, ref. 1) are frequently present. With this information as a guide, the following model was constructed and used herein.

The linear wave present is assumed to be a single plane wave. The nonlinear disturbance is assumed to consist of an infinite set of plane waves, which are periodic in both space and time and propagate in the direction of the linear component. The order of magnitude of the ascending coefficients of the series is monotonically decreasing. A perturbation technique (ref. 1) that can readily incorporate these assumptions into the nonlinear equations is used. With the model so defined, it is possible to calculate the relative magnitude of the nonlinear effects.

The theory is developed for plasma waves in an unbounded medium propagating both perpendicular and parallel to a superimposed steady-state magnetic field.

A similar analysis has been made (ref. 2) for waves propagating perpendicular to a magnetic field immersed in an electron gas that has a uniform neutralizing background of infinitely heavy ions. In the present calculation, however, the effects of the cooperative motion of singularly charged ions are included; in addition, the case of waves propagating parallel along the magnetic field is investigated. Calculations of the amplitudes of the first perturbation of the linearized wave in atomic hydrogen are presented for frequencies near the cyclotron conditions for magnetic fields of 10^2, 10^3, and 10^4 gauss and plasma densities of 10^{11}, 10^{12}, and 10^{14} centimeter^{-3}. The results are also presented in nondimensional form.

SYMBOLS

a_1 — $\frac{4\pi c}{i\omega_1}$

a_2 — $\frac{4\pi c}{i\omega_2}$

B — perturbed magnetic field strength, G

B_0 — steady-state magnetic field strength, G

c — velocity of light, 2.99793×10^{10} cm/sec

D — $2\left(\frac{c}{e}\right)^2 \frac{m_e}{n_0}$

E — electric field strength, esu/cm

$\mathcal{E}$ — $\frac{|\tilde{E}_2|}{|\tilde{E}_1|^2}$

e — electronic charge, 4.80286×10^{-10} esu

f_z $\quad \dfrac{4\pi c^2 k_{1,z}}{\alpha B_0 \omega_1^2}\left(-iE_{1,y} + \dfrac{\omega_1}{\omega_{c,i}} E_{1,x}\right)$

g_z $\quad \dfrac{4\pi c^2 k_{1,z}}{\alpha B_0 \omega_1^2}\left(iE_{1,x} + \dfrac{\omega_1}{\omega_{c,i}} E_{1,y}\right)$

J electric current density, abamp/cm^2

K nondimensional wave number, kc/ω

k wave number, cm^{-1}

m particle mass, g

n charge-particle density, cm^{-3}

$\overline{P}_{e,i}$ momentum transfer

q particle electric charge, esu

S $\quad s_x J_{1,x}^2 + t_x J_{1,y}^2 + u_x J_{1,y} J_{1,x} + x_x J_{1,x} + y_x J_{1,y}$

s_x $\quad \dfrac{8\pi i c k_1 B_0}{\rho_0 \omega_p^2 \omega_{c,i}}\left[\Omega_{II} - \dfrac{\omega_2}{\alpha}\left(\dfrac{1}{\omega_1} + \dfrac{1}{\omega_2}\right)\right]$

s_y $\quad \dfrac{8\pi c k_1 B_0}{\rho_0 \omega_p^2}\left[\Omega_{II}\left(\dfrac{1}{\omega_1} + \dfrac{1}{\omega_2}\right) - \dfrac{\omega_2}{\alpha \omega_{c,i}^2}\right]$

T $\quad s_y J_{1,x}^2 + t_y J_{1,y}^2 + u_y J_{1,y} J_{i,x} + x_y J_{1,x} + y_y J_{1,y}$

t time, sec

t_x $\quad \dfrac{4\pi i c k_1 B_0 \omega_2}{\rho_0 \omega_1 \omega_{c,i} \alpha}\left(\dfrac{2}{\omega_p^2} + \dfrac{1}{\alpha \omega_1^2}\right)$

t_y $\quad -\dfrac{4\pi c k_1 B_0 \Omega_{II}}{\rho_0 \omega_1}\left(\dfrac{2}{\omega_p^2} + \dfrac{1}{\alpha \omega_1^2}\right)$

u_x $\quad \dfrac{4\pi c^2 k_1 m_i}{\rho_0 \omega_p^2 e}\left[-\Omega_{II}\dfrac{\omega_{c,i}}{\omega_1}\left(4 + 2\dfrac{\omega_1}{\omega_2} + \dfrac{\omega_{c,e}\omega_{c,i}}{\omega_1^2}\right) + 2\dfrac{\omega_2}{\alpha \omega_{c,i}}\right]$

u_y $\quad \dfrac{4\pi i c^2 k_1 m_i}{\rho_0 \omega_{pe}^2}\left[2\Omega_{II} - \dfrac{\omega_2}{\omega_1}\dfrac{1}{\alpha}\left(4 + 2\dfrac{\omega_1}{\omega_2} + \dfrac{\omega_{c,e}\omega_{c,i}}{\omega_1^2}\right)\right]$

v $\quad$ velocity (fluid velocity if not subscripted), cm/sec

W $\quad$ nondimensional frequency, $\dfrac{\omega}{\omega_{c,i}}$

W_e $\quad$ nondimensional electron cyclotron frequency, $\dfrac{\omega_{c,e}}{\omega_{c,i}}$

W_p $\quad$ nondimensional plasma frequency, $\dfrac{\omega_p}{\omega_{c,i}}$

x_x $\quad \dfrac{k_1}{\omega_1}\left[\left(\dfrac{c}{n_0 e} E_{1,x} - \dfrac{iB_0}{\rho_0 \omega_1} E_{1,y}\right)\Omega_{II} - \dfrac{B_0 E_{1,x}}{\alpha \omega_{c,i}\rho_0}\right]$

x_y $\quad \dfrac{ik_1}{\omega_1}\left[- \dfrac{B_0}{\omega_2 \rho_0} E_{1,x}\Omega_{II} + \dfrac{\omega_2}{\alpha \omega_{c,i}}\left(\dfrac{c}{n_0 e} E_{1,x} - \dfrac{iB_0}{\rho_0 \omega_1} E_{1,y}\right)\right]$

y_x $\quad \dfrac{k_1}{\omega_1}\left[\left(\dfrac{iB_0}{\rho_0 \omega_1} E_{1,x} + \dfrac{c}{n_0 e} E_{1,y}\right)\Omega_{II} - \dfrac{B_0 E_{1,y}}{\rho_0 \alpha \omega_{c,i}}\right]$

y_y $\quad i\dfrac{k_1}{\omega_1}\left[- \dfrac{B_0}{\rho_0 \omega_2} E_{1,y}\Omega_{II} + \left(\dfrac{\omega_2}{\alpha \omega_{c,i}}\right)\left(\dfrac{iB_0}{\rho_0 \omega_1} E_{1,x} + \dfrac{c}{n_0 e} E_{1,y}\right)\right]$

α $\quad \dfrac{4\pi\rho c^2}{B_0^2}$

β $\quad \dfrac{n^{1/2}}{B_0}$

Δ_1 $\quad a_1\left[\Omega_I^2 - \left(\dfrac{\omega_1}{\alpha \omega_{c,i}}\right)^2\right]$

Δ_2	$a_2\left[\Omega_{II}^2 - \left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)^2\right]$
ρ	mass density, g/cm^3
ρ_e	charge density, esu/cm^3
Ψ	stress tensor
Ω_I	$-\frac{\omega_1^2}{\omega_p^2} + \frac{1}{\alpha}$
Ω_{II}	$-\frac{\omega_2^2}{\omega_p^2} + \frac{1}{\alpha}$
Ω_{III}	$1 + \frac{\Omega_{II}}{\Omega_{IV}}$
Ω_{IV}	$\Omega_{II}^2 - \left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)^2$
ω	frequency, rad/sec
ω_c	cyclotron frequency, $\frac{B_0 e}{mc}$
ω_p	plasma frequency, $\left(\frac{4\pi n e^2}{m_e}\right)^{1/2}$
ω_I	lower hybrid frequency, $\left[\omega_{c,i}^{-1}\omega_{c,e}^{-1} + \left(\omega_{p,i}^2 + \omega_{c,i}^2\right)^{-1}\right]^{-1/2}$
ω_{II}	upper hybrid frequency, $\left(\omega_{c,e}^2 + \omega_{p,e}^2\right)^{1/2}$
ω_{III}	$\left(\omega_{c,e}\omega_{c,i} + \omega_{p,e}^2\right)^{1/2}$

Subscripts:

e	electron
i	ion

x, y, z	Cartesian components
0	steady-state conditions
1, 2, and 3	order of perturbation
$\perp$	perpendicular to magnetic field
$\|\|$	parallel to magnetic field

Superscripts:

$\bar{\ }$	vector quantity
$\sim$	factor of quantity independent of space and time
$\hat{\ }$	unit vector

THEORY

The major characteristics of plasma waves can be determined from the dispersion relation that relates the wave number to the forcing frequency. The dispersion relation of a magnetoplasma can be derived with the use of Ohm's law and the equation of motion for a fluid element of ionized gas. These two expressions, however, are generally not referenced in a convenient form that includes the nonlinear terms. In order to obtain these terms, Ohm's law and the equation of motion are derived starting with the hydromagnetic equations (ref. 3, eq. 2-4) of the various charged species present in the plasma. For the ions, this expression is

$$n_i m_i\left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_i \cdot \nabla \bar{v}_i\right) = n_i e\left(\bar{E} + \frac{\bar{v}_i \times \bar{B}}{c}\right) - \nabla \cdot \bar{\Psi}_i + \bar{P}_{e,i} \tag{1}$$

and for the electrons, it is

$$n_e m_e\left(\frac{\partial \bar{v}_e}{\partial t} + \bar{v}_e \cdot \nabla \bar{v}_e\right) = -n_e e\left(\bar{E} + \frac{\bar{v}_e \times \bar{B}}{c}\right) - \nabla \cdot \bar{\Psi}_e + \bar{P}_{e,i} \tag{2}$$

Adding equations (1) and (2) while making use of the conservation of mass $\partial\rho/\partial t + \nabla \cdot \rho\bar{v} = 0$ results in the following form of the equation of motion:

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho\bar{v} \cdot \nabla\bar{v} + \left(\frac{c}{e}\right)^2 \frac{m_e}{n} (\bar{J} \cdot \nabla\bar{J} + \bar{J}\nabla \cdot \bar{J}) = \bar{J} \times \bar{B} + \rho_e \bar{E} \tag{3}$$

When equations (1) and (2) are multiplied by e/cm_e and e/cm_i and are then subtracted, the following form of Ohm's law results:

$$\frac{m_e c}{e^2} \frac{\partial \bar{J}}{\partial t} + \frac{m_e}{e} \nabla \cdot (n_i\bar{v}_i\bar{v}_i - n_e\bar{v}_e\bar{v}_e) = n_e\left(1 + \frac{m_e}{m_i}\right)\bar{E} + \frac{m_e}{m_i}\frac{\rho_e}{e}\bar{E} + \frac{\rho}{m_i c}\bar{v} \times \bar{B} - \frac{1}{e}\left(1 - \frac{m_e}{m_i}\right)\bar{J} \times \bar{B} \tag{4}$$

where $\bar{v}$ and $\bar{J}$ are defined by

$$\rho\bar{v} = n_e m_e \bar{v}_e + n_i m_i \bar{v}_i \tag{5}$$

and

$$\frac{c}{e}\bar{J} = n_i\bar{v}_i - n_e\bar{v}_e \tag{6}$$

Since the plasma is a cold and collisionless gas, the stress tensor and momentum transfer term have been neglected in the derivation of equations (3) and (4). It is convenient to derive the nonlinear effects for the perpendicular propagating wave ($\bar{k}$ perpendicular to $\bar{B}_0$) separately from those of the parallel propagating wave ($\bar{k}$ parallel to $\bar{B}_0$).

Perpendicular Propagation ($\bar{k}$ Perpendicular to $\bar{B}_0$)

To specify the form of the solution, it is first assumed that the running wave of the fundamental along with any higher or lower harmonics all propagate perpendicular to the magnetic field; that is, $\bar{k}$ perpendicular to $\bar{B}_0$. Further, it is assumed that any forced disturbance shall be composed of traveling waves of the form

$$\bar{A} = \sum_{j=1}^{\infty} \bar{A}_j = \sum_{j=1}^{\infty} \tilde{\bar{A}}_j e^{i(k_j x + \omega_j t)} \tag{7}$$

where $j = 1$ is the linear solution of equations (3) and (4). The components of the series for $j > 1$ originate from the nonlinear terms in equations (3) and (4). If an additional assumption is made stating that terms in ascending order of j represent terms of decreasing orders of magnitude, it becomes possible to form a set of recurrence equations. If the zero-order values of ρ and $\overline{B}$ are constant while those of $\overline{J}$, ρ_e, $\overline{v}$, and $\overline{E}$ vanish, the first-order terms of equations (3) and (4) reduce to

$$\rho_0 \frac{\partial \overline{v}_1}{\partial t} = \overline{J}_1 \times \overline{B}_0 \tag{8}$$

and

$$\frac{m_e c}{e^2 n} \frac{\partial \overline{J}_1}{\partial t} = E_1 + \frac{\overline{v}_1 \times \overline{B}_0}{c} - \frac{1}{ne} \overline{J}_1 \times \overline{B}_0 \tag{9}$$

respectively.

The assumption of an exponential solution (eq. (7)) permits gradients and time derivatives to be expressed as $ik\hat{x}$ and $i\omega$, respectively, so that $\nabla \cdot \overline{A}$ and $\partial\overline{A}/\partial t$ now becomes $\sum_{j=1}^{\infty} ik_j A_{j,x}$ and $\sum_{j=1}^{\infty} i\omega_j \overline{A}_j$, respectively. These equivalences are used throughout the the following derivations. From the conservation of mass,

$$\rho_1 = -\rho_0 \frac{k_1}{\omega_1} v_{1,x} \tag{10}$$

Consequently, the second-order component of equation (3) is

$$\rho_0 \frac{\partial \overline{v}_2}{\partial t} - \rho_0 \frac{k_1}{\omega_1} v_{1,x} \frac{\partial \overline{v}_1}{\partial t} + \rho_0 \overline{v}_1 \cdot \nabla \overline{v}_1 + \left(\frac{c}{e}\right)^2 \frac{m_e}{n_0} (\overline{J}_1 \cdot \nabla \overline{J}_1 + \overline{J}_1 \nabla \cdot \overline{J}_1)$$

$$= \overline{J}_2 \times \overline{B}_0 + \overline{J}_1 \times \overline{B}_1 + \rho_{1,e} \overline{E}_1 \tag{11}$$

The charge density is expressed as

$$\rho_{1,e} = e(n_{1,i} - n_{1,e}) \tag{12}$$

but since

$$n_{1,i} = -\frac{k_1}{\omega_1} n_0 v_{1,x} + \frac{m_e}{\rho}\frac{c}{e} J_{1,x} \tag{13}$$

and

$$n_{1,e} = -\frac{k_1}{\omega_1} n_0 v_{1,x} - \frac{m_i}{\rho}\frac{c}{e} J_{1,x} \tag{14}$$

which are obtained from the conservation of the mass of the species, $\rho_{1,e}$ may be written as

$$\rho_{1,e} = -\frac{k_1}{\omega_1} cJ_{1,x} \tag{15}$$

Substituting equation (15) along with the identities

$$\bar{v}_1 \cdot \nabla\bar{v}_1 = v_{1,x} ik_1 \bar{v}_1 \tag{16}$$

and

$$\bar{J}_1 \cdot \nabla\bar{J}_1 + \bar{J}_1 \nabla \cdot \bar{J}_1 = 2J_{1,x} ik_1 \bar{J}_1 \tag{17}$$

into equation (11) results in

$$\bar{v}_2 = -2\frac{c}{e}^2 \frac{m_e}{n_0} \frac{k_1}{\rho_0\omega_2} J_{1,x}\bar{J}_1 - \frac{i}{\rho_0\omega_2} \bar{J}_2 \times \bar{B}_0 - \frac{i}{\rho_0\omega_2} \bar{J}_1 \times \bar{B}_1 - \frac{k_1}{\rho_0\omega_2}\frac{c}{i\omega_1} J_{1,x}\bar{E}_1 \tag{18}$$

When equations (14) and (15) are used, the second-order equation obtained from Ohm's law reduces to

$$
\begin{aligned}
\frac{m_e c}{e^2 n_0} \frac{\partial \bar{J}_2}{\partial t} + \frac{m_e}{e} \nabla \cdot (\bar{v}_{1,i}\bar{v}_{1,i} - \bar{v}_{1,e}\bar{v}_{1,e}) = \bar{E}_2 - \frac{k_1}{\omega_1}\left(v_{1,x} - \frac{m_i}{\rho}\frac{c}{e} J_{1,x}\right)\bar{E}_1 \\
- \frac{m_e}{m_i}\frac{k_1}{\omega_1}\frac{c}{n_0 e} J_{1,x}\bar{E}_1 + \frac{\bar{v}_1 \times \bar{B}_1}{c} + \frac{\bar{v}_2 \times \bar{B}_0}{c} \\
- \frac{k_1 v_{1,x}}{\omega_1 c}\bar{v}_1 \times \bar{B}_0 - \frac{1}{n_0 e}\bar{J}_1 \times \bar{B}_1 - \frac{1}{n_0 e}\bar{J}_2 \times \bar{B}_0 \qquad (19)
\end{aligned}
$$

The spatial derivative term of equation (19) can be replaced by

$$
\nabla \cdot (\bar{v}_{1,i}\bar{v}_{1,i} - \bar{v}_{1,e}\bar{v}_{1,e}) = \frac{2ick_1}{n_0 e}\left(\bar{v}_1 J_{1,x} + \bar{J}_1 v_{1,x} - \frac{m_i c}{\rho e} J_{1,x}\bar{J}_1\right) \qquad (20)
$$

Expressions for $\bar{v}_1$ and $\bar{v}_2$ are given by equations (8) and (18), respectively. From Faraday's law, $\bar{B}_1$ can be represented as

$$
\bar{B}_1 = \frac{ck_1}{\omega_1}(E_{1,z}\hat{y} - E_{1,y}\hat{z}) \qquad (21)
$$

When these substitutions are made, equation (19) becomes

$$
\begin{aligned}
\frac{m_e c}{e^2 n_0} i\omega_2 \bar{J}_2 - \frac{iB_0^2}{\rho_0 \omega_2 c}\bar{J}_{2,\perp} - \bar{E}_2 = \frac{m_e}{e}\frac{2ick_1}{n_0 e}\left(\frac{iB_0}{\rho_0\omega_1}(-J_{1,x}\hat{y} + \bar{J}_{1,y}\hat{x})J_{1,x} + \frac{iB_0}{\rho_0\omega_1}J_{1,y}\bar{J}_1 + \frac{m_i c}{\rho e}J_{1,x}\bar{J}_1\right) + \frac{ik_1 B_0}{\rho_0\omega_1^2}J_{1,y}\bar{E}_1 + \frac{k_1 m_i c}{\omega_1 \rho e}J_{1,x}\bar{E}_1 \\
- \frac{i}{\rho_0\omega_1^2} k_1 B_0(\bar{J}_{1,\perp}E_{1,y} + J_{1,y}E_{1,z}\hat{z}) - \frac{2cm_e k_1 B_0}{e^2 n_0 \omega_2 \rho_0}\left[J_{1,x}(-J_{1,x}\hat{y} + J_{1,y}\hat{x})\right] \\
- \frac{ik_1 B_0 E_{1,y}}{\rho_0\omega_2\omega_1}\bar{J}_{1,\perp} - \frac{ik_1 E_{1,z}J_{1,z}B_0}{\rho_0\omega_2\omega_1}\hat{y} + \frac{ik_1 J_{1,x}B_0}{\omega_2\omega_1\rho_0}(-E_{1,x}\hat{y} + E_{1,y}\hat{x}) - \frac{k_1 B_0^3 J_{1,y}}{\omega_1^3 c\rho_0^2}\bar{J}_{1,\perp} \\
- \frac{1}{n_0 e}\frac{ck_1}{\omega_1}\left[E_{1,y}(J_{1,x}\hat{y} - J_{1,y}\hat{x}) + E_{1,z}(J_{1,x}\hat{z} - J_{1,z}\hat{x})\right] - \frac{B_0}{n_0 e}(-J_{2,x}\hat{y} + J_{2,y}\hat{x}) \qquad (22)
\end{aligned}
$$

In order to compare the amplitudes of the nonlinear fields with those of the linear fields, it is necessary to first express $\overline{J}_2$ as a function of $\overline{E}_1$ and $\overline{E}_2$. This operation is accomplished by substituting the following expressions for $\overline{J}_1$ (obtained from eqs. (8) and (9)) into equation (22):

$$J_{1,x} = \frac{E_{1,x}\Omega_I - E_{1,y}\dfrac{i\omega_1}{\alpha\omega_{c,i}}}{\Delta_1} \tag{23a}$$

$$J_{1,y} = \frac{E_{1,y}\Omega_I + E_{1,x}\dfrac{i\omega_1}{\alpha\omega_{c,i}}}{\Delta_1} \tag{23b}$$

$$J_{1,z} = \frac{\omega_p^2}{ci\omega_1 4\pi} E_{1,z} \tag{23c}$$

Before proceeding, it will be convenient at this point in the derivation to split the problem into two cases. These two cases arise since there are two dispersion relations for the linear wave equation. The wave equation can be expressed as

$$\begin{pmatrix} \dfrac{\omega_1^2}{c^2} & 0 & 0 \\ 0 & \dfrac{\omega_1^2}{c^2} - k_1^2 & 0 \\ 0 & 0 & \dfrac{\omega_1^2}{c^2} - k_1^2 \end{pmatrix} \begin{pmatrix} E_{1,x} \\ E_{1,y} \\ E_{1,z} \end{pmatrix} - \frac{\omega_1}{c} i4\pi \begin{pmatrix} J_{1,x} \\ J_{1,y} \\ J_{1,z} \end{pmatrix} = 0 \tag{24}$$

Since $\overline{J}_1$ is a function of $\overline{E}_1$, equation (24) represents a set of three homogeneous equations. For perpendicular propagating waves ($\overline{k}$ perpendicular to $\overline{B}_0$), $E_{1,z}$ is not coupled to $E_{1,x}$ or $E_{1,y}$. Consequently, two independent forms of the dispersion relation exist. The wave consisting only of $E_{1,z}$ is called the ordinary wave, while the remaining wave consisting only of $E_{\perp}$ is the extraordinary wave. The dispersion relation for the ordinary wave is

$$k_1^2 = \frac{\omega_1^2}{c^2} - \frac{\omega_p^2}{c^2} \tag{25}$$

while for the extraordinary wave it is

$$k_1^2 = \frac{\omega_1^2}{c^2} \frac{\left(\omega_1^2 - \omega_1\omega_{c,e} - \omega_{III}^2\right)\left(\omega_1^2 + \omega_1\omega_{c,e} - \omega_{III}^2\right)}{\left(\omega_1^2 - \omega_I^2\right)\left(\omega_1^2 - \omega_{II}^2\right)} \tag{26}$$

These expressions are identical to those presented in reference 4. Since both waves are independent, the higher-order effects will be developed first for the extraordinary wave and then for the ordinary wave.

<u>Extraordinary wave ($\bar{k}$ perpendicular to $\bar{B}_0$, $\bar{E}_1$ perpendicular to $\bar{B}_0$).</u> - If the linear currents expressed by equation (23) are substituted into equation (22), the following expressions can be obtained for the first harmonic currents when $E_{1,z} = 0$:

$$J_{2,x} = \frac{\Omega_{II}E_{2,x} - \dfrac{i\omega_2}{\alpha\omega_{c,i}}E_{2,y} + S(J_{1,x}, J_{1,y})}{\Delta_2} \tag{27a}$$

$$J_{2,y} = \frac{\Omega_{II}E_{2,y} + \dfrac{i\omega_2}{\alpha\omega_{c,i}}E_{2,x} + T(J_{1,x}, J_{1,y})}{\Delta_2} \tag{27b}$$

$$J_{2,z} = \frac{\omega_p^2}{i4\pi c\omega_2} E_{2,z} \tag{27c}$$

where

$$S = s_x J_{1,x}^2 + t_x J_{1,y}^2 + u_x J_{1,y} J_{1,x} + x_x J_{1,x} + y_x J_{1,y} \tag{28}$$

and

$$T = s_y J_{1,x}^2 + t_y J_{1,y}^2 + u_y J_{1,y} J_{1,x} + x_y J_{1,x} + y_y J_{1,y} \tag{29}$$

Substituting the expressions for $\overline{J}_2$ into the wave equation of the $\tilde{\overline{A}}_2 e^{i(k_2 x+\omega_2 t)}$ wave gives

$$\begin{pmatrix} \frac{\omega_2^2}{c^2} & 0 & 0 \\ 0 & \frac{\omega_2^2}{c^2} - k_2^2 & 0 \\ 0 & 0 & \frac{\omega_2^2}{c^2} - k_2^2 \end{pmatrix} \begin{pmatrix} E_{2,x} \\ E_{2,y} \\ E_{2,z} \end{pmatrix} -14\pi \frac{\omega_2}{c} \begin{pmatrix} J_{2,x} \\ J_{2,y} \\ J_{2,y} \end{pmatrix} = 0 \tag{30}$$

This substitution results in a set of nonhomogeneous equations that can be solved for the electric fields. Hence,

$$E_{2,x} = \tilde{E}_{2,x} e^{i(k_2 x+\omega_2 t)} = \frac{-\frac{1}{\Omega_{IV}} \left[S\left(\Omega_{III} - k_2^2 \frac{c^2}{\omega_2^2}\right) + iT \frac{\left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)}{\Omega_{IV}} \right] e^{i(2k_1 x+2\omega_1 t)}}{\Omega_{III}\left(\Omega_{III} - \frac{k_2^2 c^2}{\omega_2^2}\right) - \frac{\left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)^2}{\Omega_{IV}^2}} \tag{31a}$$

$$E_{2,y} = \tilde{E}_{2,y} e^{i(k_2 x + \omega_2 t)} = i\Omega_{IV}\left(\frac{\alpha\omega_{c,i}}{\omega_2}\right)\left(\Omega_{III}\tilde{E}_{2,x} + \frac{S}{\Omega_{IV}}\right)e^{i(2k_1 x + 2\omega_1 t)} \tag{31b}$$

$$\left(\frac{\omega_2^2}{c^2} - k_2^2 - \frac{\omega_p^2}{c^2}\right)E_{2,z} = 0 \tag{31c}$$

Both $S/\tilde{E}_{1,x}^2$ and $T/\tilde{E}_{1,x}^2$ are proportional to B_0^{-1}; in addition, they are functions of W_1, W_p, and W_e only, where

$$W = \frac{\omega}{\omega_{c,i}}$$

$$W_e = \frac{\omega_{c,e}}{\omega_{c,i}}$$

$$W_p = \frac{\omega_p}{\omega_{c,i}}$$

Consequently, these terms can be written as

$$\frac{S}{\tilde{E}_{1,x}^2} = \frac{1}{B_0} S(W, W_p, W_e)$$

and

$$\frac{T}{\tilde{E}_{1,x}^2} = \frac{1}{B_0} T(W, W_p, W_e)$$

Equations (31a) and (31b) can thus be nondimensionalized when multiplying through by $B_0/\tilde{E}_{1,x}^2$, which results in

$$\frac{\tilde{E}_{2,x}B_0}{\tilde{E}^2_{1,x}} = \frac{-\frac{1}{\Omega_{IV}}\left[S(\Omega_{III} - K_1^2) + iT\,\frac{W_2W_e}{W_p^2\Omega_{IV}}\right]}{\Omega_{III}(\Omega_{III} - K_1^2) - \left(\frac{W_2W_e}{W_p^2\Omega_{IV}}\right)^2} \tag{32a}$$

and

$$\frac{\tilde{E}_{2,y}B_0}{\tilde{E}^2_{1,x}} = -\frac{\Omega_{IV}W_p^2}{W_eW_2}\left(\Omega_{III}\,\frac{\tilde{E}_{2,x}B_0}{E^2_{1,x}} + \frac{S}{\Omega_{IV}}\right) \tag{32b}$$

where $K(= kc/\omega)$, Ω_{III}, and Ω_{IV} are nondimensional expressions and functions of W, W_p, and W_e only. Since $\tilde{E}_{2,x}B_0/\tilde{E}^2_{1,x}$ and $\tilde{E}_{2,y}B_0/\tilde{E}^2_{1,x}$ are invarient to any changes in ω_1, n, B_0, and m_i where W_1, W_p, and W_e remain constant, they are convenient forms in which to present the results.

These expressions for the various second-order electric fields are functions of x and t. Since the amplitudes of the various harmonics are constant, the only way that equations (31a) and (31b) may be satisfied for all x and t is for

$$\left.\begin{aligned} k_2 &= 2k_1 \\ \omega_2 &= 2\omega_1 \end{aligned}\right\} \tag{33}$$

The phase velocity, consequently, is identical for both $\overline{E}_2$ and $\overline{E}_1$. Substituting these values of k_2 and ω_2 into equation (31c) dictates that $E_{2,z}$ must vanish. Another possible solution exists for equation (31c). If the wave represented by $E_{2,x}$ and $E_{2,y}$ is uncoupled from $E_{2,z}$, a separate wave number for $E_{2,z}$ can be determined from (31c). However, the resulting dispersion relation is just that for the ordinary wave obtained from linearized theory. Since it is not coupled to the nonlinear wave and since it is specified that the fundamental is an extraordinary wave, $E_{2,z}$ can again be set equal to zero. The electric fields associated with the second-order perturbation, consequently, are perpendicular to $\overline{B}_0$ and are functions of the superimposed steady-state magnetic field. In addition, since all disturbances propagate transverse to the magnetic field, the first harmonic may be called an extraordinary wave.

In summary, when an extraordinary wave exists in a plasma, the nonlinearity of the equations requires an additional extraordinary wave. This nonlinear component has the same phase velocity but twice the frequency as the linear component.

The wave amplitude, that is,

$$\left[(\Re e E_{2,x})^2 + (\Re e E_{2,y})^2\right]^{1/2}$$

or

$$|\tilde{E}_2| \propto \left(|\tilde{E}_{2,x}|^2 + |\tilde{E}_{2,y}|^2\right)^{1/2} \tag{34}$$

is probably of more interest than the expressions for the nonlinear components of the disturbance. This equation is still inconvenient because of the terms $E_{1,x}$ and $E_{1,y}$. However, $E_{1,y}$ is directly proportional to $E_{1,x}$ (see eqs. (23) and (24)) so that a factor of $E_{1,x}^2$ may be extracted from equations (31a) and (31b) and hence cancelled from the resulting expression. If the right hand side of equation (34) is divided by

$$|\tilde{E}_{1,x}|^2 + |\tilde{E}_{1,y}|^2 \tag{35}$$

which is proportional to $|\tilde{E}_1|^2$, there results

$$\mathcal{E}_{\perp} = \frac{\left(\dfrac{|\tilde{E}_{2,x}|^2}{|\tilde{E}_{1,x}|^4} + \dfrac{|\tilde{E}_{2,y}|^2}{|\tilde{E}_{1,x}|^4}\right)^{1/2}}{1 + \dfrac{|\tilde{E}_{1,y}|^2}{|\tilde{E}_{1,x}|^2}} \tag{36}$$

which is an expression independent of $E_{1,x}$ and $E_{1,y}$. Both $|\tilde{E}_2|$ and $|\tilde{E}_1|$ represent the time average of the wave amplitude. Equation (36) will hereinafter be referred to as the "amplitude ratio" of the extraordinary wave. It is the ratio of the amplitude of the nonlinear wave to the square of the amplitude of the linear wave. A nondimensional wave amplitude ratio $|\tilde{E}_2|B_0/|\tilde{E}_1|^2$ will also be investigated. The description of the behavior of the amplitude ratios will be discussed in the section RESULTS.

Ordinary wave ($\overline{k}$ perpendicular to $\overline{B}_0$, $\overline{E}_1$ parallel to $\overline{B}_0$). - In this case, $E_{1,x} = E_{1,y} = 0$ while $E_{1,z} \neq 0$. In an approach similar to that used with the extraordinary wave, plasma currents may be determined for the ordinary waves. Thus, for $E_{1,y} = E_{1,x} = 0$ and $E_{1,z} \neq 0$,

$$J_{2,x} = \frac{i\Omega_{II}E_{2,x} + \dfrac{\omega_2}{\alpha\omega_{c,i}} E_{2,y} + \dfrac{ik_1c}{n_0e\omega_1}\left(\Omega_{II} - \dfrac{1}{\alpha}\right)E_{1,z}J_{1,z}}{\dfrac{4\pi c}{\omega_2}\Omega_{IV}} \tag{37a}$$

$$J_{2,y} = \frac{i\Omega_{II}E_{2,y} - \dfrac{\omega_2}{\alpha\omega_{c,i}} E_{2,x} + \dfrac{k_1e}{\omega_1n_0e}\left(\Omega_{II}\dfrac{\omega_{c,i}}{\omega_2} - \dfrac{\omega_2}{\alpha\omega_{c,i}}\right)E_{1,z}J_{1,z}}{\dfrac{4\pi c}{\omega_2}\Omega_{IV}} \tag{37b}$$

$$J_{2,z} = -i\frac{\omega_p^2}{4\pi c\omega_2}E_{2,z} \tag{37c}$$

After substituting equation (37) into equation (30), the following is obtained:

$$\tilde{E}_{2,x} = \frac{\dfrac{k_1}{4\pi\omega_1^2}\dfrac{\tilde{E}_{1,z}^2}{\Omega_{IV}}\left[-\dfrac{\omega_2^2}{n_0e}\left(\Omega_{III} - \dfrac{k_2^2c^2}{\omega_2^2}\right) - \dfrac{\left(\dfrac{\omega_2}{\alpha\omega_{c,i}}\right)}{\Omega_{IV}}\dfrac{\omega_p^2B_0}{cn_0\omega_2m_i}\left(\dfrac{\omega_2^2m_i}{n_0e^24\pi} - \dfrac{1}{\alpha}\right)\right]}{\Omega_{III}\left(\Omega_{III} - \dfrac{k_2^2c^2}{\omega_2^2}\right) - \dfrac{\left(\dfrac{\omega_2}{\alpha\omega_{c,i}}\right)^2}{\Omega_{IV}^2}} \tag{38a}$$

$$\tilde{E}_{2,y} = \frac{-\dfrac{k_1}{4\pi\omega_1^2}\dfrac{\tilde{E}_{1,z}}{\Omega_{IV}}\left[\left(\Omega_{III}\dfrac{\omega_p^2B_0}{n_0\omega_2cm_i}\,\dfrac{\omega_2^2m_i}{n_0e^24\pi} - \dfrac{1}{\alpha}\right) + \dfrac{\left(\dfrac{\omega_2}{\alpha\omega_{c,i}}\right)\omega_2^2}{\Omega_{IV}n_0e}\right]}{\Omega_{III}\left(\Omega_{III} - k_2^2\dfrac{c^2}{\omega_2^2}\right) - \dfrac{\left(\dfrac{\omega_2}{\alpha\omega_{c,i}}\right)^2}{\Omega_{IV}^2}} \tag{38b}$$

$$\left(\frac{\omega_2^2}{c^2} - k_2^2 - \frac{\omega_p^2}{c^2}\right)E_{2,z} = 0 \tag{38c}$$

Equations (38a) and (38b) can be nondimensionalized by multiplying through by $B_0/\tilde{E}_{1,z}^2$ and by expressing the variables in terms of W_1, W_p, and W_e. This results in

$$\frac{\tilde{E}_{2,x}B_0}{\tilde{E}_{1,z}^2} = \frac{-\dfrac{K_1W_e}{\Omega_{IV}W_1W_p^2}\left[W_2^2(\Omega_{III} - K_2^2) + \dfrac{W_e^2}{W_p^2\Omega_{IV}}(W_2^2 - 1)\right]}{\Omega_{III}\left(\Omega_{III} - K_2^2\right) - \left(\dfrac{W_2W_e}{W_p^2\Omega_{IV}}\right)^2} \tag{39a}$$

and

$$\frac{\tilde{E}_{2,y}B_0}{\tilde{E}_{1,z}^2} = \frac{-\dfrac{K_1W_e}{\Omega_{IV}W_1W_p^2}\left[\dfrac{\Omega_{III}}{W_2}(W_2^2 - 1) + \dfrac{W_2^3}{W_p^2}\right]}{\Omega_{III}\left(\Omega_{III} - K_2^2\right) - \left(\dfrac{W_2W_e}{W_p^2\Omega_{IV}}\right)^2} \tag{39b}$$

Since $\tilde{E}_{2,x}B_0/\tilde{E}_{1,z}^2$ and $\tilde{E}_{2,y}B_0/\tilde{E}_{1,z}^2$ are invarient to any change in ω_1, n, B_0, and m_i where W_1, W_p, and W_e remain constant, they represent a convenient form in which to present the results.

These second-order electric fields are analogous to the fields of the extraordinary fundamental, since both depend on the magnetic field and since both have x and y electric field components but no z component. Consequently, the first harmonic associated with an ordinary wave fundamental can be called an extraordinary wave. The propagation constant k_2 and the frequency ω_2 can be shown to be equal to $2k_1$ and $2\omega_1$ in a fashion similar to that used with the fundamental extraordinary plasma wave. Substituting these values of k_2 and ω_2 into equation (38c) dictates that a coupled $E_{2,z}$ wave must vanish. An uncoupled wave $E_{2,z}$ cannot exist for the same reason as was presented in the previous section. The amplitude ratio for the ordinary wave, consequently, is

$$\mathscr{E}_\perp = \frac{\left(|\tilde{E}_{2,x}|^2 + |\tilde{E}_{2,y}|^2\right)^{1/2}}{|\tilde{E}_{1,z}|^2} \tag{40}$$

which is proportional to $|\tilde{E}_2|/|\tilde{E}_{1,z}|^2$. A nondimensional wave amplitude ratio $|\tilde{E}_2|B_0/|\tilde{E}_1|^2$ will also be investigated. The description of the behavior of the amplitude ratios will be discussed in the section RESULTS.

Parallel Propagation ($\bar{k}$ Parallel to $\bar{B}_0$, $\bar{E}_1$ Perpendicular to $\bar{B}_0$)

When $\bar{k}$ is parallel to $\bar{B}_0$, equation (7) is not valid; in this case,

$$\bar{A} = \sum_{j=1}^{\infty} \tilde{\tilde{A}}_j e^{i(k_{j,z}z+\omega_j t)} \tag{41}$$

Since the propagation vector $\bar{k}$ is now directed axially, all relations involving $\bar{k}$ and its spatial derivative must be changed. The derivation of the amplitude ratio is presented with equations analogous to those used with perpendicularly propagating waves. With the use of

$$\nabla \cdot \bar{v}_1 = ik_{1,z}v_{1,z}$$

the zero-order equation of motion becomes

$$\rho_1 = -\rho_0 \frac{k_{1,z}}{\omega_1} v_{1,z} \tag{42}$$

while the second-order part of the equation of motion becomes

$$\rho_0 i\omega_2 \bar{v}_2 + Dik_{1,z}J_{1,z}\bar{J}_1 = \bar{J}_2 \times \bar{B}_0 + \bar{J}_1 \times \bar{B}_1 - \frac{k_{1,z}}{\omega_1} cJ_{1,z}\bar{E}_1 \tag{43}$$

When equation (12) and

$$n_{1,i} = - \frac{k_{1,z} n_0}{\omega_1} \left(v_{1,z} + \frac{m_e}{\rho} \frac{c}{e} J_{1,z} \right) \tag{44}$$

and

$$n_{1,e} = - \frac{k_{1,z} n_0}{\omega_1} \left(v_{1,z} - \frac{m_i}{\rho} \frac{c}{e} J_{1,z} \right) \tag{45}$$

are used, the charge density may be written as

$$\rho_{1,e} = - \frac{k_{1,z}}{\omega_1} c J_{1,z} \tag{46}$$

Substituting equation (46) along with

$$\bar{v}_1 \cdot \nabla \bar{v}_1 = i k_{1,z} v_{1,z} \bar{v}_1 \tag{47}$$

and

$$\bar{J}_1 \cdot \nabla \bar{J}_1 = \bar{J}_1 \nabla \cdot \bar{J}_1 = i k_{1,z} J_{1,z} \bar{J}_1 \tag{48}$$

into equation (43) results in

$$\bar{v}_2 = \frac{-D k_{1,z} J_{1,z} \bar{J}_1}{\rho_0 \omega_2} - \frac{i \bar{J}_2 \times \bar{B}_0}{\rho_0 \omega_2} - \frac{i \bar{J}_1 \times \bar{B}_1}{\rho_0 \omega_2} + \frac{i k_{1,z} c J_{1,z} \bar{E}_1}{\rho_0 \omega_1 \omega_2} \tag{49}$$

When equations (45) and (46) are used, the second-order equation obtained from Ohm's law reduces to

$$\frac{m_e c}{e^2 n_0} \frac{\partial \bar{J}_2}{\partial t} + \frac{m_e}{e} \nabla \cdot \left(\bar{v}_{1,i} \bar{v}_{1,i} - \bar{v}_{1,e} \bar{v}_{1,e} \right) = \bar{E}_2 - \frac{k_{1,z}}{\omega_1} \left(v_{1,z} - \frac{m_i}{\rho_0 e} c J_{1,z} \right) \bar{E}_1$$

$$- \frac{m_e}{m_i} \frac{k_{1,z} c}{n_0 e \omega_1} J_{1,z} \bar{E}_1 + \frac{\bar{v}_1 \times \bar{B}_1}{c} + \frac{\bar{v}_2 \times \bar{B}_0}{c} - \frac{\rho_0 k_{1,z} v_{1,z}}{n_0 m_i c \omega_1} \bar{v}_1 \times \bar{B}_0 - \frac{\bar{J}_1 \times \bar{B}_1}{n_0 e} - \frac{\bar{J}_2 \times \bar{B}_0}{n_0 e} \tag{50}$$

The spatial derivative term can be replaced by

$$\nabla \cdot (\overline{v}_{1,i}\overline{v}_{1,i} - \overline{v}_{1,e}\overline{v}_{1,e}) = \frac{2icm_i k_{1,z}}{\rho_0 e}\left(J_{1,z}\overline{v}_1 + v_{1,z}\overline{J}_1 - \frac{m_i c}{\rho_0 e} J_{1,z}\overline{J}_1\right) \tag{51}$$

Expressions for $\overline{v}_1$ and $\overline{v}_2$ are given in equations (8) and (49), respectively. From Faraday's law, $\overline{B}_1$ can be represented as

$$\overline{B}_1 = \frac{ck_{1,z}}{\omega_1}(E_{1,y}\hat{x} - E_{1,x}\hat{y}) \tag{52}$$

When these substitutions are made, equation (50) becomes

$$\begin{aligned}
&\frac{im_e c\omega_2}{e^2 n_0}\overline{J}_2 + \frac{m_e}{e}\left(\frac{2ick_{1,z}}{n_0 e}\right)\left(J_{1,z}\overline{v}_1 - \frac{m_i c}{\rho_0 e}J_{1,z}\overline{J}_1\right) = \overline{E}_2 \\
&\qquad + \frac{ck_{1,z}m_i}{\rho_0 e\omega_1}J_{1,z}\overline{E}_1 + \frac{\overline{v}_1 \times \overline{B}_1}{c} - \frac{Dk_{1,z}B_0}{c\rho_0\omega_2}J_{1,z}(-J_{1,x}\hat{y} + J_{1,y}\hat{x}) \\
&\qquad + \frac{iB_0^2}{c\rho_0\omega_2}\overline{J}_{2,\perp} - \frac{ik_{1,z}B_0}{\rho_0\omega_2\omega_1}(-J_{1,z}E_{1,x}\hat{y} + J_{1,z}E_{1,y}\hat{x}) \\
&\qquad + \frac{ik_{1,z}B_0}{\rho_0\omega_1\omega_2}J_{1,z}(E_{1,y}\hat{x} - E_{1,x}\hat{y}) - \frac{\overline{J}_1 \times \overline{B}_1}{n_0 e} - \frac{\overline{J}_2 \times \overline{B}_0}{n_0 e}
\end{aligned} \tag{53}$$

where $v_{1,z}$ was set equal to zero. The first-order equation of motion given by equation (8) forces $v_{1,z}$ to vanish.

In order to compare the amplitudes of the nonlinear fields with those of the linear fields, $\overline{J}_2$ must first be expressed as a function of $\overline{E}_1$ and $\overline{E}_2$. This is accomplished by substituting the expressions for $\overline{J}_1$ (eqs. (23)) into equation (53), which is the second-order Ohm's law.

Before proceeding, it will be convenient to split the problem into two cases as was done for the perpendicular propagation. These two cases arise since there are two dispersion relations for the linear wave equation. The wave equation can be expressed as

$$\begin{pmatrix} \frac{\omega_1}{c^2} - k_1^2 & 0 & 0 \\ 0 & \frac{\omega_1^2}{c^2} - k_1^2 & 0 \\ 0 & 0 & \frac{\omega_1^2}{c^2} \end{pmatrix} \begin{pmatrix} E_{1,x} \\ E_{1,y} \\ E_{1,z} \end{pmatrix} - \frac{4\pi i \omega_1}{c} \begin{pmatrix} J_{1,x} \\ J_{1,y} \\ J_{1,z} \end{pmatrix} = 0 \tag{54}$$

Since $\overline{J}_1$ is a linear function of $\overline{E}_1$, equation (54) represents a set of three homogeneous equations. For purely parallel propagating waves $E_{1,x}$ and $E_{1,y}$ are not coupled to $E_{1,z}$. The dispersion relation for the two possible transverse waves referred to as right and left circular waves are

$$k_1^2 = \frac{\omega_1^2}{c^2} \left[1 - \frac{\omega_p^2}{(\omega_1 + \omega_{c,i})(\omega_1 - \omega_{c,e})} \right] \tag{55}$$

and

$$k_1^2 = \frac{\omega_1^2}{c^2} \left[1 - \frac{\omega_p^2}{(\omega_1 - \omega_{c,i})(\omega_1 + \omega_{c,e})} \right] \tag{56}$$

These expressions are identical to those presented in reference 4. Since both waves are independent, the higher-order effects for these two waves will be developed separately.

The linear currents expressed by equation (23) can be substituted into equation (53) since the form of the equations are independent of $\overline{k}$ (no space derivatives were involved in their derivation). Consequently, they can be used with the various modes of propagation. When $E_{1,z} = 0$, the following expression for the first harmonic current is obtained:

$$J_{2,x} = \frac{\left(-\frac{\omega_2^2}{\omega_p^2} + \frac{1}{\alpha}\right)E_{2,x} - \frac{i\omega_2}{\alpha\omega_{c,i}} E_{2,y}}{a_2\left[\left(-\frac{\omega_2^2}{\omega_p^2} + \frac{1}{\alpha}\right)^2 - \left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)^2\right]} \tag{57a}$$

$$J_{2,y} = \frac{\left(-\frac{\omega_2^2}{\omega_p^2} + \frac{1}{\alpha}\right)E_{2,y} + \frac{i\omega_2}{\alpha\omega_{c,i}} E_{2,x}}{a_2\left[\left(-\frac{\omega_2^2}{\omega_p^2} + \frac{1}{\alpha}\right)^2 - \left(\frac{\omega_2}{\alpha\omega_{c,i}}\right)^2\right]} \tag{57b}$$

$$J_{2,z} = -\frac{i\omega_p^2}{4\pi c\omega_2}\left(E_{2,z} + f_z J_{1,x} + g_z J_{1,y}\right) \tag{57c}$$

Substituting equations (23), (53), and (57a), (57b), and (57c) into the wave equation

$$\begin{pmatrix} \frac{\omega_2^2}{c^2} - k_2^2 & 0 & 0 \\ 0 & \frac{\omega_2^2}{c^2} - k_2^2 & 0 \\ 0 & 0 & \frac{\omega^2}{c^2} \end{pmatrix}\begin{pmatrix} E_{2,x} \\ E_{2,y} \\ E_{2,z} \end{pmatrix} - \frac{4\pi i\omega_2}{c}\begin{pmatrix} J_{2,x} \\ J_{2,y} \\ J_{2,z} \end{pmatrix} = 0 \tag{58}$$

results in a set of three equations. Two of these expressions couple $E_{2,x}$ and $E_{2,y}$:

$$\left(\frac{\omega_2^2}{c^2} - k_2^2 - \frac{\omega_2}{c}\frac{4\pi i}{\Delta_2}\,\Omega_{II}\right)E_{2,x} + \frac{4\pi i}{\Delta_2}\left(\frac{\omega_2}{c}\right)\left(\frac{i\omega_2}{\alpha\omega_{c,i}}\right)E_{2,y} = 0 \tag{59a}$$

$$-\frac{\omega_2}{c}\left(\frac{4\pi i}{\Delta_2}\right)\left(\frac{i\omega_2}{\alpha\omega_{c,i}}\right)E_{2,x} + \left[\frac{\omega_2^2}{c^2} - k_2^2 - \left(\frac{\omega_2}{c}\right)\left(\frac{4\pi i}{\Delta_2}\right)\Omega_{II}\right]E_{2,y} = 0 \tag{59b}$$

$$E_{2,z} = \frac{4\pi k_{1,z}\omega_p^2}{\left(\frac{\omega_2^2}{c^2} - \frac{\omega_p^2}{c^2}\right)\alpha B_0\omega_1^2}\left[\left(-iE_{1,y} + \frac{\omega_1}{\omega_{c,i}}E_{1,x}\right)J_{1,x} + \left(iE_{1,x} + \frac{\omega_1}{\omega_{c,i}}E_{1,y}\right)J_{1,y}\right] \tag{59c}$$

Equation (59c) can be nondimensionalized by multiplying through by $B_0/\tilde{E}_{1,x}^2$ and expressing the variables in terms of W_1, W_p, and W_e, which results in

$$\frac{\tilde{E}_{2,z}B_0}{\tilde{E}_{1,x}^2} = \frac{K_{1,z}W_e}{\left(W_2^2 - W_p^2\right)}\left[\left(-i\frac{\tilde{E}_{1,y}}{\tilde{E}_{1,x}} + W_1\right)\left(\frac{4\pi c}{\omega_{c,i}}\frac{\tilde{J}_{1,x}}{\tilde{E}_{1,x}}\right) + \left(i + W_1\frac{\tilde{E}_{1,y}}{\tilde{E}_{1,x}}\right)\left(\frac{4\pi c}{\omega_{c,i}}\frac{\tilde{J}_{1,x}}{\tilde{E}_{1,x}}\right)\right] \tag{60}$$

Since $(4\pi c/\omega_{c,i})(\tilde{J}_{1,x}/\tilde{E}_{1,x})$ is invarient to any change in ω_1, n, B_0, and m_i when W_1, W_p, and W_e remain constant, $\tilde{E}_{2,z}B_0/\tilde{E}_{1,x}^2$ represents a convenient form in which to present the results.

The relation between k_2 and k_1 is found as was done for the perpendicular wave by equating those factors that are spatially and time dependent. This procedure yields equation (33). The wave represented by equation (59c) is coupled to the fundamental wave; however, the waves represented by equations (59a) and (59b) are not. The terms $E_{2,x}$ and $E_{2,y}$ must vanish since k_2 is determined from equation (33) and does not satisfy equation (59a) and (59b) unless they do. Another possible solution exists for equations (59a) and (59b). If the $E_{2,z}$ wave is uncoupled from the $E_{2,\perp}$ wave, a separate wave number for $E_{2,\perp}$ can be determined from (59a) and (59b). However, the resulting dispersion relation

$$k_2^2 = \frac{\omega_2^2}{c^2}\left[1 - \frac{\omega_p^2}{(\omega_2 \mp \omega_{c,i})(\omega_2 \pm \omega_{c,e})}\right] \tag{61}$$

is identical to the relation for the fundamental that is assumed present. Consequently, no additional information is obtained from (59a) and (59b). The electric field associated with the second-order perturbation consequently lies along the magnetic field and is actually a function of the superimposed steady-state magnetic field. This latter characteristic is unusual since, generally, when a wave is a function of E_z, it does not depend on the magnetic field.

In summary, when a parallel propagating linear wave ($\overline{k}$ parallel to $\overline{B}_0$, $\overline{E}_1$ perpendicular to $\overline{B}_0$) exists in a plasma, the nonlinearity of the equation requires that an additional wave be formed whose electric field is perpendicular to that of the linear ordinary wave ($\overline{E}_2$ parallel to $\overline{B}_0$).

As with the perpendicular wave, an amplitude ratio is obtained by dividing the second-order perturbation by the square of the amplitude of the fundamental. Hence, the amplitude ratio is

$$\mathscr{E}_{||} = \frac{\dfrac{|\widetilde{E}_{2,z}|}{|\widetilde{E}_{1,x}|^2}}{\left(1 + \dfrac{|\widetilde{E}_{1,y}|^2}{|\widetilde{E}_{1,x}|^2}\right)} \tag{62}$$

A nondimensional amplitude ratio $|\widetilde{E}_{2,z}|B_0/|\widetilde{E}_1|^2$ will also be investigated. The description of the behavior of the amplitude ratios will be discussed in the section RESULTS.

RESULTS

In the previous section, general characteristics of the nonlinear wave perturbation were noted. The nonlinearity of the equations requires that additional waves are formed. For both an ordinary and extraordinary first-order wave, the second-order nonlinear disturbance is an extraordinary wave. The magnitudes of higher-order disturbances were not examined.

To determine the relative magnitude of the second-order wave, or the amplitude ratio, numerical solutions of the appropriate equations developed in the previous section were obtained. The various cases will be discussed individually, but first a few general comments are appropriate. For both the ordinary and the right and left circularly polarized fundamental, the linear approximation is satisfactory even for large amplitude linear components. The disturbances associated with the extraordinary fundamental are small above a certain critical frequency; below this frequency, they can be large.

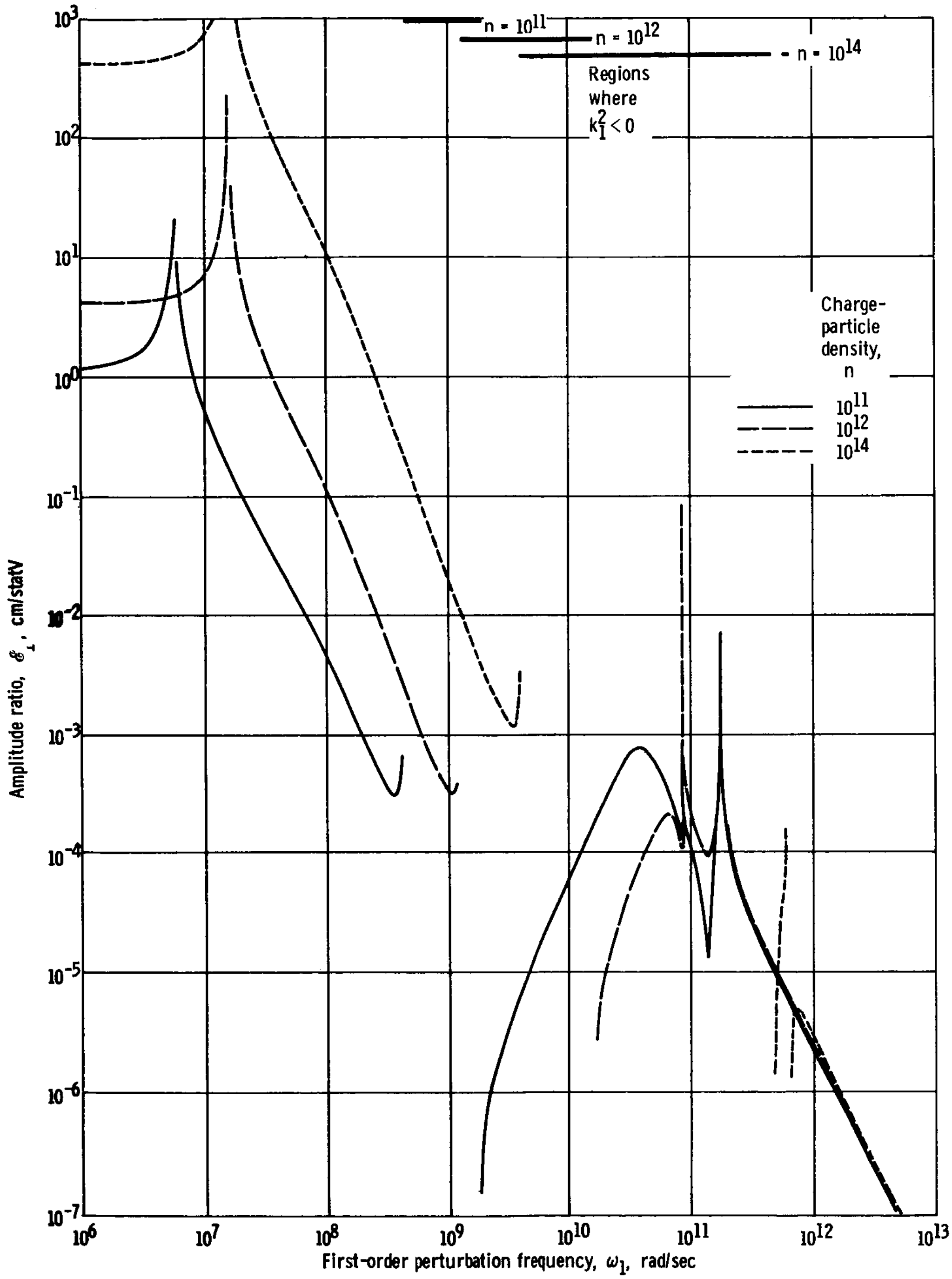
The quantity of pragmatic interest is not so much the magnitude of the various field components but the amplitude of the total electric field. Consequently, the amplitude ratio $\mathscr{E}$ of the nonlinear wave is presented in all plots of this report. The nondimensional amplitude ratio is also presented since it can compactly represent many n and B_0 combinations. The horizontal bars appearing at the top of the plots represent the frequency range where the square of the wave number of the fundamental wave is negative and consequently where the assumed periodicity does not exist.

To obtain the ratio of the amplitudes of the first harmonic to that of the fundamental from the figures, $\mathscr{E}$ is multiplied by $|E_1|$, where $|E_1|$ is the amplitude of the fundamental expressed in statvolts per centimeter. To obtain the amplitude of the first harmonic, $\mathscr{E}$ is multiplied by $|E_1|^2$. The regions of the plots of the amplitude ratio where nonlinear terms are important will vary with the magnitude of the fundamental. Such regions can be defined by the conditions that $\mathscr{E} > 0.1|E_1|^{-1}$; this condition represents a situation where the amplitude of the first harmonic is 1/10 that of the fundamental. This condition, however, is where the perturbation technique becomes questionable. Consequently, the numerical value of the nonlinear amplitude will be questionable. Even though the numerical value is in doubt, these regions represent a condition for which nonlinear effects are important.

Perpendicular Propagation

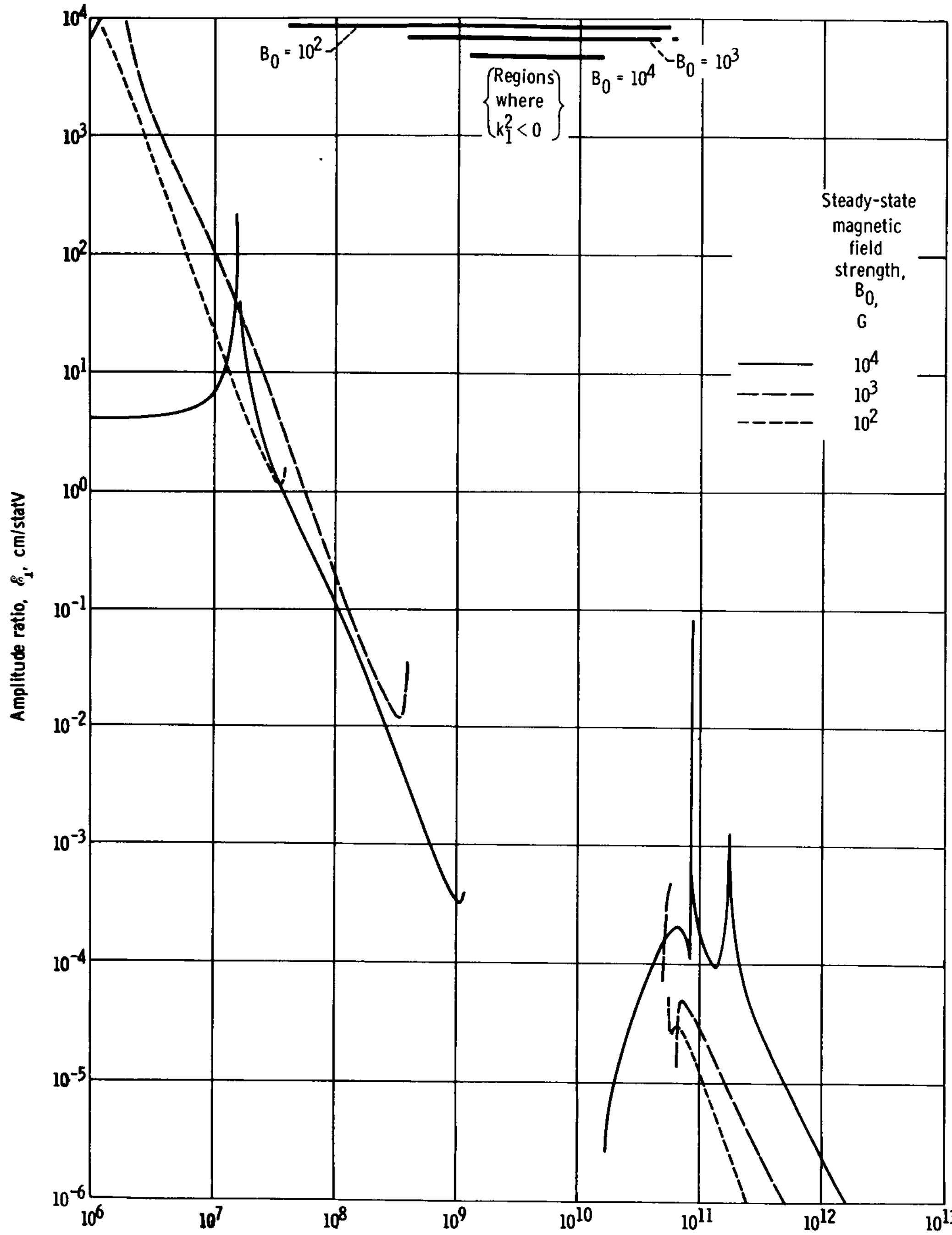
Extraordinary wave. - Computer solutions of the amplitude ratio of the extraordinary fundamental represented by equation (36) are plotted in figure 1(a) as functions of ω_1 for the field intensities of 10^4 gauss and for densities of 10^{11}, 10^{12}, and 10^{14} centimeters^{-3}. (Since it is inconvenient to use nondimensional amplitude ratios in order to establish n and B trends, dimensional plots shall be used.) Four singularities are present on this plot. The range of influence of all but the first is limited to rather restricted frequency regions. At frequencies below the first singularity, the amplitude ratio of the first harmonic wave is fairly constant and decreases as the density increases. The amplitude ratio for a field strength of 10^2, 10^3, and 10^4 gauss and a density of 10^{12} centimeter^{-3} is presented in figure 1(b). Decreasing the field intensity has shifted the first singular point to lower frequencies. Concomitantly, the amplitude ratio for frequencies less than the first singularity increases significantly as the field decreases.

The singularity at the lowest frequency occurs when the denominator of the expression for the nonlinear field contribution, that is, $|\tilde{E}_2|$ (eq. (31)), vanishes. At low frequencies, this occurs at the Alfvén velocity $\omega = kc/\alpha^{1/2}$ (ref. 1, eq. (3-29)). For an atomic hydrogen plasma and for the range of β's calculated, the first singularity occurs at about 0.2 of the ion cyclotron frequency (fig. 2). At the second singularity $\mathscr{E}$ increases because of an increasing k_1 $(k \rightarrow \infty)$; this corresponds to the "lower hybrid" resonance $(\omega = \omega_I)$ of the extraordinary fundamental. The third singularity



(a) Steady-state magnetic field strength, 10^4 gauss

Figure 1. - Amplitude ratio $\mathscr{E}_\perp$ of extraordinary wave in atomic hydrogen as function of frequency.



(b) Charge-particle density, 10^{12} centimeter^{-3}

Figure 1. - Concluded.

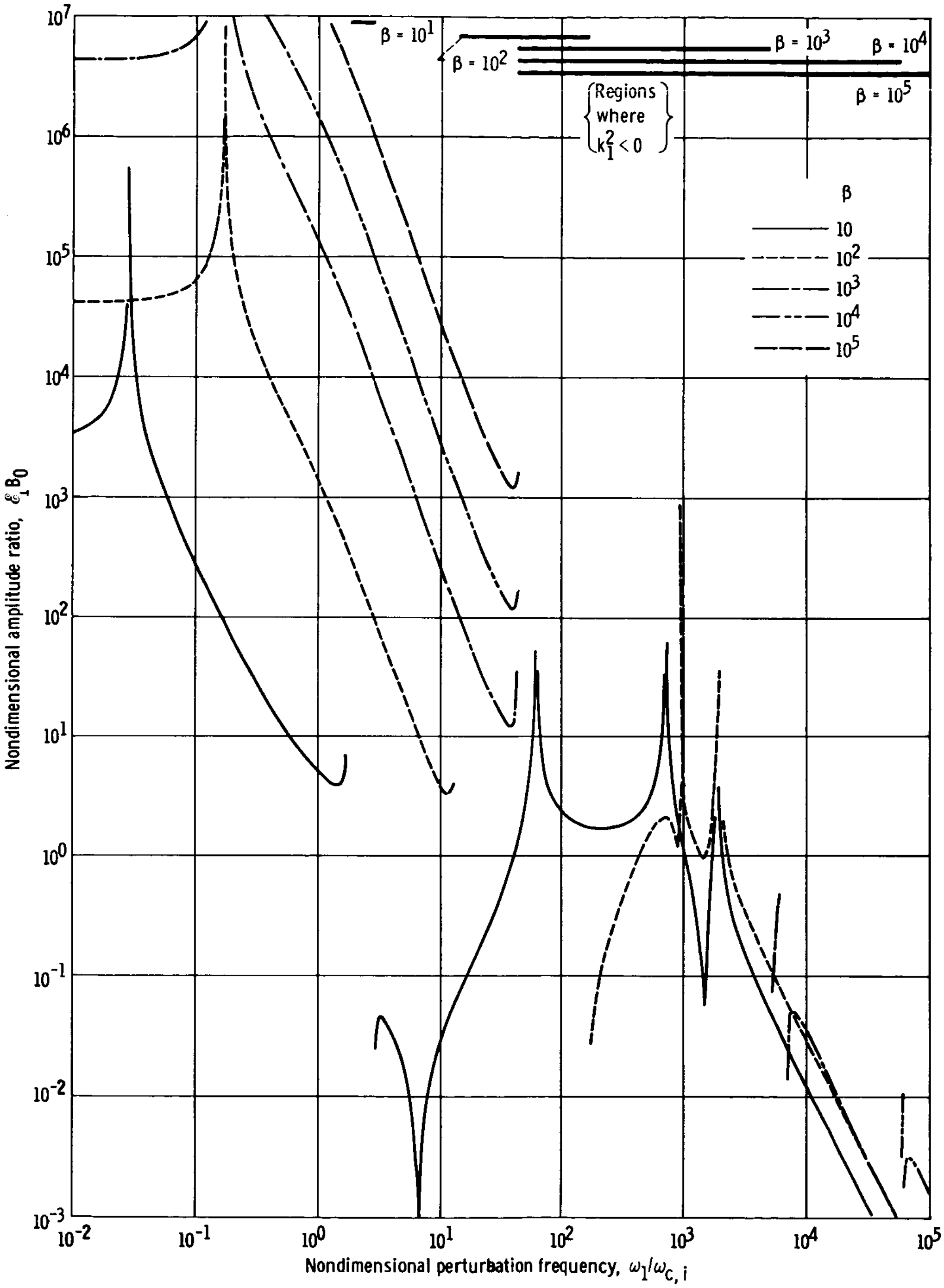


Figure 2. - Nondimensional amplitude ratio of extraordinary wave as function of nondimensional frequency. Nondimensional electron cyclotron frequency, 1836. 12.

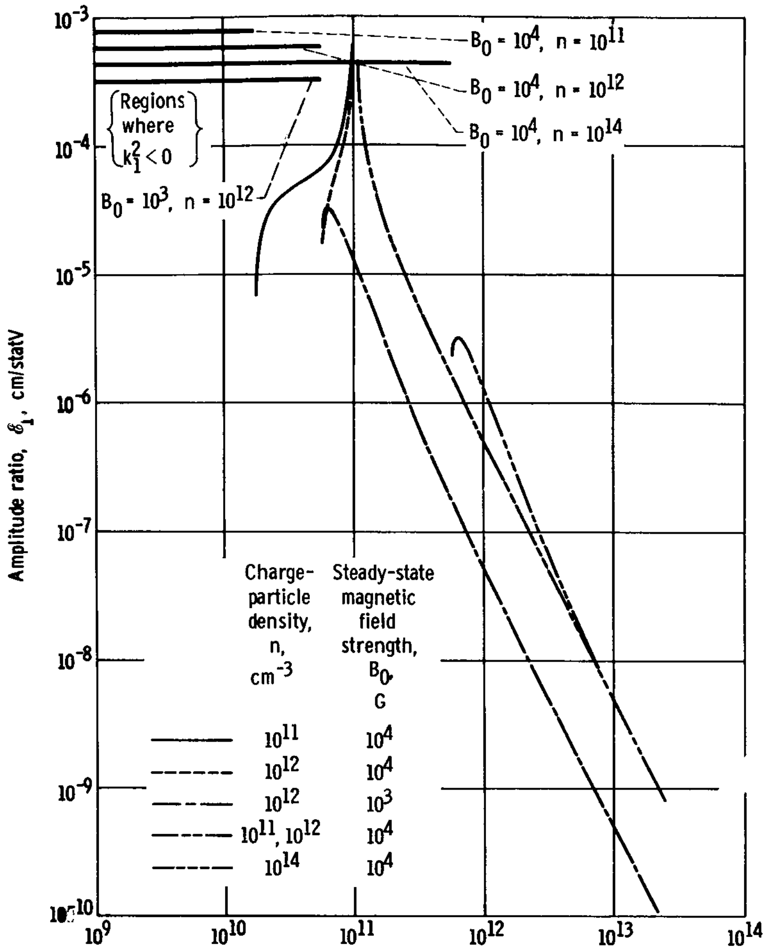


Figure 3. - Amplitude ratio $\mathscr{E}_1$ of ordinary wave in atomic hydrogen as function of frequency.

occurs when $\bar{J}_2$ increases without limit. This result occurs when $\omega_2 \approx \omega_{c,e}$; presumably there exists a subharmonic of the electron cyclotron resonance. When $\omega_1 \approx \omega_{c,e}/2$, there exists a nonlinear component with a frequency of $\omega_{c,e}$. This component is the one which resonates at $\omega_{c,e}$. The fourth peak consists of two closely spaced singularities, the first of which occurs at the upper hybrid frequency $\omega = \omega_{II}$ of the extraordinary fundamental. These two poles approached each other as the density increased so that at 10^{14} centimeter^{-3} they could not be numerically resolved.

These singularities and those found for the other cases suggest that in plasmas where particle or wave resonance $k = \infty$ exist, such as in experiments of particle energy addition or in wave damping in magnetic beaches, nonlinear effects are important. The nonlinear effects at these resonances could be realized of course only if dissipative phenomena were negligible. This assumption is probably very unrealistic at conditions of particle and wave resonance. Consequently, it is expected that the spires appearing

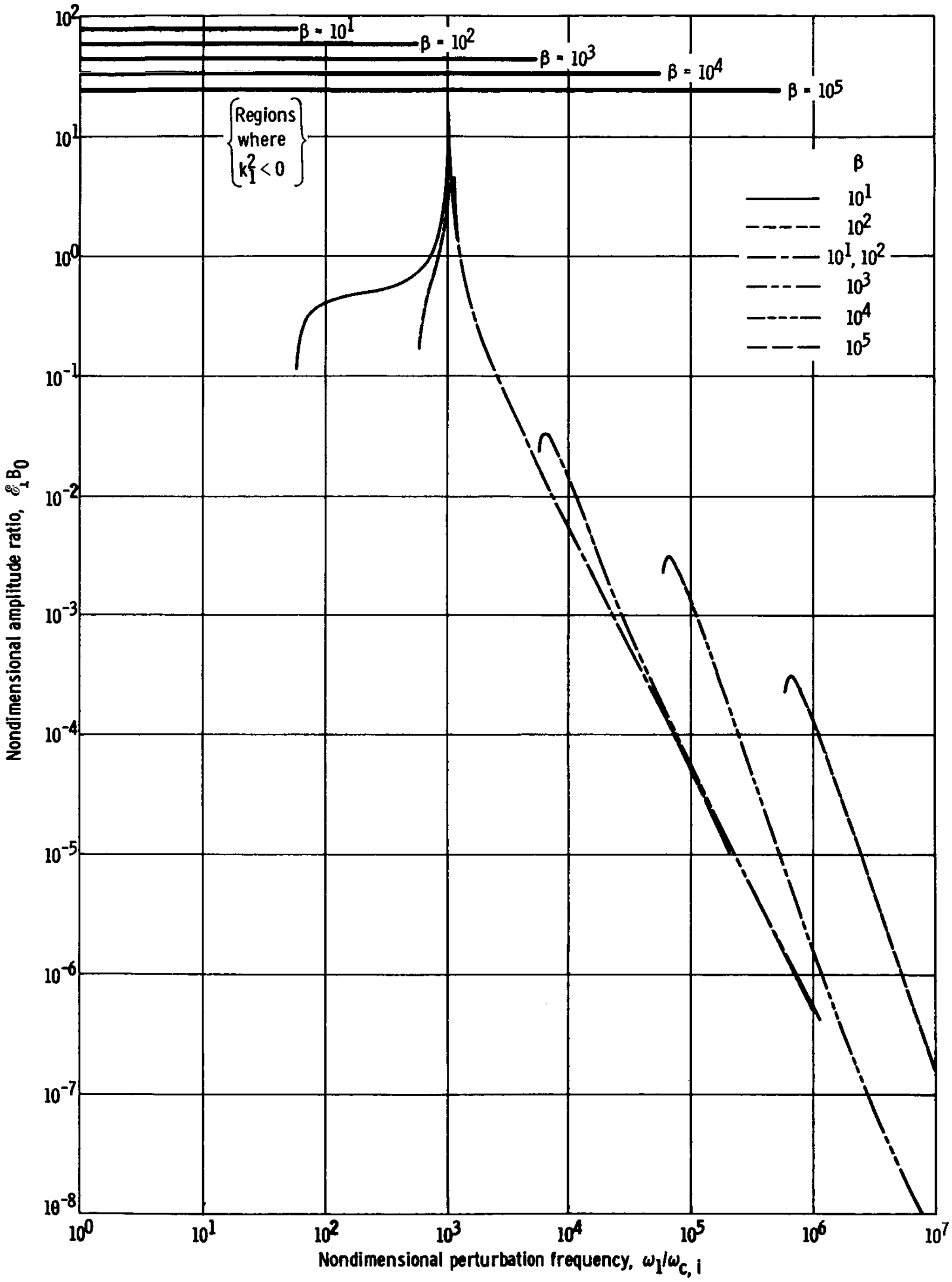


Figure 4. - Nondimensional amplitude ratio of ordinary wave as function of nondimensional frequency. Nondimensional electron cyclotron frequency, 1836.12.

in $\mathscr{E}$ would be greatly modified. This conclusion suggests that investigations of dissipative wave phenomena should include the nonlinear terms of equations (2) and (3). At nonresonant conditions, dissipative effects would have much less influence on wave propagation. Consequently, nonlinearity can be evaluated at these conditions with much more confidence. The nondimensional amplitude ratio for the extraordinary wave is plotted in figure 2. Instead of using W_p as a parameter, a proportional quantity defined as $\beta = n^{1/2}/B_0$ is used.

Ordinary wave. - Computer calculations of the amplitude ratio for the second-order perturbation of an ordinary wave (eq. (40)) are plotted in figure 3. As in the previous case, this perturbation is also an extraordinary wave. There is, however, an essential difference not only in the shape of the curves but more significantly in the magnitude of the amplitude ratio. The amplitude of the second-order perturbation originating from an ordinary wave is considerably smaller than the amplitude originating from an extraordinary wave. The amplitude ratio of this nonlinear effect of the ordinary wave is plotted in figure 3 as a function of frequency for magnetic field intensities of 10^3 and 10^4 gauss and particle densities of 10^{11}, 10^{12}, and 10^{14} centimeter^{-3}. For the conditions plotted, the amplitude ratio is very small compared with that for the case of the extraordinary wave, except for an extremely narrow frequency band. The half breadth can be so narrow that it may not be of any practical interest. As an example, at 10^4 gauss and 10^{12} centimeter^{-3}, the nonlinear amplitude is greater than the linear component for a fundamental of 1 statvolt per centimeter only between the frequencies of 1.05437×10^{11} and 1.05442×10^{11} second^{-1}, a condition that is beyond the accuracy of the model. The nondimensional amplitude ratio for the ordinary wave is plotted in figure 4.

Parallel Propagation

Calculations of the nonlinear perturbation for the right and left circularly polarized parallel propagating wave (eq. (62)) are not presented in figure form since their magnitudes are quite small. Only in the near vicinity of the singularities is equation (62) significant; the half breadths are so narrow that they are not of practical interest in the present analysis since the breadths could easily be masked by dissipative effects. At 10^4 gauss and 10^{12} centimeter^{-3}, the nonlinear amplitude is greater than the linear component for a fundamental of 1 statvolt per centimeter only between the frequencies of 9.5794521×10^7 and 9.5794566×10^7 second^{-1}, a condition which is beyond the accuracy of the model.

CONCLUSIONS

If second-order terms are included in the equation of motion and Ohm's law for a magnetoplasma and if a small amplitude periodic variation in space and time is assumed, waves originate from the nonlinear terms of the equations. These waves are present along with the normal linear perturbations in plasmas. The second-order perturbations or harmonic waves have the following characteristics:

1. The first harmonic, if assumed to propagate parallel to $\bar{k}$, is an extraordinary wave regardless of whether the fundamental is ordinary or extraordinary. In both cases it will have the same phase velocity as the fundamental with twice the frequency.

2. For an extraordinary wave in an atomic hydrogen plasma, the first harmonic of the nonlinear effects is characterized by four rather narrow singularities. Another region of large nonlinear effects occurs at frequencies lower than the first singularity, which occurs at about 0.2 of the ion cyclotron frequency. The nonlinear effect in this region increases when the density increases and/or the magnetic field decreases. Above about 0.2 of the ion cyclotron frequency, the nonlinear effect decreases as the frequency increases except in the various frequency regions where singularities occur.

3. For an ordinary wave, the amplitude of the first harmonic (propagating in the direction of the fundamental) is generally small.

4. For the right and left circularly polarized parallel propagating waves, the nonlinear effects are negligible except for a very narrow range of frequencies and for a large amplitude fundamental.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 28, 1966,
129-01-05-09-22.

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